Local Volume Estimate for Manifolds with L^2-bounded Curvature

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Abstract: we obtain a local volume growth for complete, noncompact Riemannian manifolds with small integral bounds and with Bach tensor having finite L^2 norm in dimension 4.

1 Introduction

It is important to study asymptotic behavior of complete manifold without the assumption of pointwise Ricci curvature bound. A volume growth and curvature decay result was obtained in [4] for various classes of complete, noncompact, Bach-flat metrics in dimension 4. Some similar results were also claimed in [1].

In this note we consider a more general case, that is, the Bach tensor may not necessarily vanish. Since Bach tensor can be viewed as a second derivative of the Ricci tensor, there will be a priori no L^p bound for it, where $p > 2$. So we may consider the case that the L^2 norm of the Bach tensor is finite. Our main result is to give a local volume estimate:

Theorem 1.1. *Let X be a complete, noncompact 4-dimensional Riemannian manifold. Let $B(p, r)$ be a geodesic ball around the point p. Assume that there holds the following local Sobolev inequality: for any open subset Ω,*

$$\|f\|^2_{L^4(\Omega)} \leq C_s(\Omega)\|\nabla f\|^2_2, \ \forall f \in C_0^\infty(\Omega).$$

Then there exist constants ε_0 and C (depending on the Sobolev constant $C_s(B(p,r))$) such that if

$$\|\mathrm{Rm}\|_{L^2(B(p,2r))} \leq \varepsilon_0$$

and $\mathrm{B} \in L^2(B(p,2r))$*, then*

$$\mathrm{Vol}(B(p,r)) \leq Cr^4.$$

If the Bach tensor does not vanish, then a direct computation shows that

$$\triangle \mathrm{Ric} = \mathrm{Rm} * \mathrm{Ric} + \mathrm{B},$$

where B is the Bach tensor. A standard argument to obtain the bound for the Ricci tensor is to use the elliptic Moser iteration for this equation. However, as we mentioned above, we can't assume $\mathrm{B} \in L^p$ for $p > 2$, for this will automatically give the regularity for the Ricci tensor. So it is not obvious to apply the elliptic Moser iteration directly, since now we consider an inhomogeneous equation.

To overcome this difficulty, we use the Ricci flow to smooth the Riemannian metric, which was first considered by Bemelmans, Min-Oo and Ruh [2]. Notice that since we only consider the local case, what matters is not the global L^2 bound on curvature but the local bound, that is, the L^2 norm of curvature on each geodesic ball of fixed radius. Also, a global heat flow will not control such a local bound. So, instead, we will use local Ricci flow , which was first used by D. Yang [5]. In that paper, a simple form of Moser iteration was applied to a local nonlinear heat equation. And we found that this argument works in our settings to obtain a pointwise local bound for the curvature tensor of the regularized metric via the local Ricci flow.

We end the introduction with a brief outline of the note. In Section 2, we will prove the Moser iteration for the local heat flow. The local existence of the Ricci flow will be discussed in Section 3. And the local bound for the curvature tensor of the regularized metric will be obtained in Section 4. Finally Theorem 1.1 will be proved in Section 5.

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2 Moser Iteration for a Local Heat Flow

Fix an open set $B_0 \subset X$ and a smooth compactly supported function $\phi \in C_0^\infty(B_0)$.

Let $g(t), 0 \leq t \leq T$, be a 1-parameter family of smooth Riemannian metrics. Let ∇ denote the covariant differentiation with respect to the metric $g(t)$ and $-\triangle$ be the corresponding Laplace-Beltrami operator. Let $A > 0$ be a constant that satisfies the standard Sobolev inequality

$$\left(\int_{B_0} f^4 \,\mathrm{d}\, V_g\right)^{\frac{1}{2}} \leq A \int_{B_0} |\nabla f|^2 \,\mathrm{d}\, V_g, \ f \in C_0^\infty(B_0),$$

with respect to each metric $g(t), 0 \leq t \leq T$.

Assume that for each $t \in [0, T]$,

$$\frac{1}{2} g_{ij}(0) \leq g_{ij}(t) \leq 2 g_{ij}(0) \ \text{ on } B_0.$$

All geodesic balls in this section are defined with respect to the metric $g(0)$, and therefore, are fixed open subsets of X, independent of t.

We want to study the following heat equation:

$$\frac{\partial f}{\partial t} \leq \phi^2(\triangle f + uf) + 2a\phi|\nabla\phi||\nabla f| + b(|\nabla\phi|^2 - \phi\triangle\phi)f, \ 0 \leq t \leq T, \tag{2.1}$$

where f and u are nonnegative functions on $B_0 \times [0, T]$, such that

$$\frac{\partial}{\partial t} \,\mathrm{d}\, V_g \leq c\phi^2 u \,\mathrm{d}\, V_g \tag{2.2}$$

and

$$\left(\int_{B_0} \phi^2 u^3\right)^{\frac{1}{3}} \leq \mu t^{-\frac{1}{3}}. \tag{2.3}$$

The following results in this section are due to D. Yang [5]. For convenience, we give the proofs below. Notice that our manifold is 4-dimensional.

Lemma 2.1. *Given* $p > 1, \ \psi \in C_0^\infty(B_0), \ f \in C^\infty(M), \ f \geq 0,$

$$\int_{B_0} |\nabla(\psi f^{\frac{p}{2}})|^2 \leq \frac{p^2}{2(p-1)} \int_{B_0} \psi^2 f^{p-1}(-\triangle f) \,\mathrm{d}\, V_g + \left(1 + \frac{1}{(p-1)^2}\right) \int_{B_0} |\nabla\psi|^2 f^p \,\mathrm{d}\, V_g.$$

Proof: Using integration by parts, we have

$$\begin{aligned}
\int |\nabla(\psi f^{\frac{p}{2}})|^2 &= -\int \psi f^{\frac{p}{2}} \triangle(\psi f^{\frac{p}{2}}) \\
&= \frac{p}{2} \int \psi^2 f^{p-1}(-\triangle f) + \int f^p |\nabla\psi|^2 - \frac{p(p-2)}{4} \int \psi^2 f^{p-2} |\nabla f|^2 \\
&= \frac{p^2}{2(p-1)} \int \psi^2 f^{p-1}(-\triangle f) + \frac{p}{2(p-1)} \int \psi^2 f^{p-1} \triangle f \\
&\quad + \int f^p |\nabla\psi|^2 - \frac{p(p-2)}{4} \int \psi^2 f^{p-2} |\nabla f|^2.
\end{aligned}$$

On the other hand, by Cauchy inequality,

$$
\begin{aligned}
\frac{p}{2(p-1)}\int \psi^2 f^{p-1}\triangle f &= -\frac{p}{2(p-1)}\int \nabla(\psi^2 f^{p-1})\nabla f \\
&= -\frac{p}{p-1}\int \psi f^{p-1}\nabla\psi\nabla f - \frac{p}{2}\int \psi^2 f^{p-2}|\nabla f|^2 \\
&\le \frac{1}{(p-1)^2}\int f^p|\nabla\psi|^2 + \frac{p^2}{4}\int \psi^2 f^{p-2}|\nabla f|^2 - \frac{p}{2}\int \psi^2 f^{p-2}|\nabla f|^2 \\
&= \frac{1}{(p-1)^2}\int f^p|\nabla\psi|^2 + \frac{p(p-2)}{4}\int \psi^2 f^{p-2}|\nabla f|^2.
\end{aligned}
$$

This proves the lemma. □

Lemma 2.2. *Suppose that f and u are nonnegative functions on $B_0\times[0,T]$ which satisfy (2.1), (2.2) and (2.3). For $p\ge p'\ge p_0>1$, we have*

$$
\frac{\partial}{\partial t}\int \phi^{2p'}f^p + \int |\nabla(\phi^{p'+1}f^{\frac{p}{2}})|^2 \le [(p'+1)^2C\|\nabla\phi\|_\infty^2 + C(p\mu)^3A^2t^{-1}]\int \phi^{2p'}f^p. \tag{2.4}
$$

Proof: Given $p\ge p'\ge p_0>1$, we combine Lemma 2.1 with (2.1) and (2.2) to obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int \phi^{2p'}f^p + 2\left(1-\frac{1}{p}\right)^2\int |\nabla(\phi^{p'+1}f^{\frac{p}{2}})|^2 \\
&\quad\le \int [\phi^2(\triangle(\phi^{2p'}f^p) + C\phi^{2p'}uf^p) + 2a\phi|\nabla\phi||\nabla(\phi^{2p'}f^p)| \\
&\qquad + b(|\nabla\phi|^2 - \phi\triangle\phi)\phi^{2p'}f^p].
\end{aligned} \tag{2.5}
$$

Now we estimate each term of the right hand side.

$$
\begin{aligned}
\int \phi^2\triangle(\phi^{2p'}f^p) &= -2\int \phi\nabla\phi\nabla(\phi^{2p'}f^p) \\
&= -\int 2\phi\nabla\phi(2p'\phi^{2p'-1}\nabla\phi\cdot f^p + p\phi^{2p'}f^{p-1}\nabla f) \\
&= -4p'\int \phi^{2p'}f^p|\nabla\phi|^2 - 2p\int \phi^{2p'+1}f^{p-1}\nabla\phi\nabla f \\
&\le 4p'\int \phi^{2p'}f^p|\nabla\phi|^2 + 2p\left(\int \phi^{2p'}|\nabla\phi|^2f^p\right)^{\frac{1}{2}}\left(\int \phi^{2p'+2}f^{p-2}|\nabla f|^2\right)^{\frac{1}{2}} \\
&\le 4p'\int \phi^{2p'}f^p|\nabla\phi|^2 + \frac{pC}{\varepsilon}\int \phi^{2p'}|\nabla\phi|^2f^p + p\,\varepsilon\int \phi^{2p'+2}f^{p-2}|\nabla f|^2.
\end{aligned}
$$

By a similar argument the remaining terms can be estimated as follows.

$$
\begin{aligned}
\int \phi|\nabla\phi||\nabla(\phi^{2p'}f^p)| \leq & 2p'\int \phi^{2p'}f^p|\nabla\phi|^2 + p\int \phi^{2p'+1}f^{p-1}|\nabla\phi||\nabla f|,\\
-\int (\phi\triangle\phi)\phi^{2p'}f^p = & \int \nabla\phi\nabla(\phi^{2p'+1}f^p)\\
= & (2p'+1)\int \phi^{2p'}|\nabla\phi|^2 f^p + p\int \phi^{2p'+1}f^{p-1}\nabla\phi\nabla f,\\
-\int \phi^{2p'+2}f^{p-1}\triangle f = & \int \nabla(\phi^{2p'+2}f^{p-1})\nabla f\\
= & (2p'+1)\int \phi^{2p'+1}f^{p-1}\nabla\phi\nabla f + (p-1)\int \phi^{2p'+2}f^{p-2}|\nabla f|^2.
\end{aligned}
$$

So it is easy to see that each term of right hand side of (2.5) has the form of

$$
\int |\nabla\phi|^2\phi^{2p'}f^p, \ \int \phi^{2p'+2}f^{p-2}|\nabla f|^2 \text{ or } \int u\phi^{2p'+2}f^p,
$$

where

$$
\begin{aligned}
\int \phi^{2p'+2}f^{p-2}|\nabla f|^2 = & \left(\frac{2}{p}\right)^2\int |\phi^{p'+1}\nabla f^{\frac{p}{2}}|^2\\
\leq & 2\left(\frac{2}{p}\right)^2\varepsilon\int |\nabla(\phi^{p'+1}f^{\frac{p}{2}})|^2 + 2\left(\frac{2}{p}\right)^2(p'+1)^2C_\varepsilon\int \phi^{2p'}f^p|\nabla\phi|^2.
\end{aligned}
$$

Notice that if ε is sufficiently small, then the first term of the right hand side can be absorbed into the left hand side of (2.5). Therefore we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\int \phi^{2p'}f^p + & 2\left(1-\frac{1}{p}\right)^2\int |\nabla(\phi^{p'+1}f^{\frac{p}{2}})|^2\\
\leq & (p'+1)^2C\int |\nabla\phi|^2\phi^{2p'}f^p + p\int u\phi^{2(p'+1)}f^p.
\end{aligned}
$$

Using Hölder, Sobolev, Cauchy inequalities, and (2.3), we see that

$$
\begin{aligned}
\int u\phi^{2(p'+1)}f^p \leq & \left(\int \phi^2u^3\right)^{\frac{1}{3}}\left(\int (\phi^{2p'}f^p)\right)^{\frac{1}{3}}\left(\int \phi^{4(p'+1)}f^{2p}\right)^{\frac{1}{3}}\\
\leq & \mu t^{-\frac{1}{3}}\left(\int (\phi^{2p'}f^p)\right)^{\frac{1}{3}}\cdot A^{\frac{2}{3}}\left(|\nabla(\phi^{p'+1})f^{\frac{p}{2}}|^2\right)^{\frac{2}{3}}\\
\leq & (\mu t^{-\frac{1}{3}})^3\varepsilon^{-\frac{1}{3}}\int \phi^{2p'}f^p + \varepsilon^{\frac{2}{3}}A^2\int |\nabla(\phi^{p'+1}f^{\frac{p}{2}})|^2.
\end{aligned}
$$

Thus,

$$\frac{\partial}{\partial t}\int \phi^{2p'} f^p + 2\left(1-\frac{1}{p}\right)^2 \int |\nabla(\phi^{p'+1} f^{\frac{p}{2}})|^2$$
$$\le (p'+1)C\int |\nabla\phi|^2\phi^{2p'} f^p + \varepsilon^{\frac{1}{3}} A^2 \int |\nabla(\phi^{p'+1} f^{\frac{p}{2}})|^2 + \varepsilon^{-\frac{1}{3}} p^3\mu^3 t^{-1}\int \phi^{2p'} f^p.$$

Choosing ε so that $\varepsilon^{\frac{2}{3}} A^2$ is sufficient small, we have

$$\frac{\partial}{\partial t}\int \phi^{2p'} f^p + \int |\nabla(\phi^{p'+1} f^{\frac{p}{2}})|^2$$
$$\le [(p'+1)^2 C\|\nabla\phi\|_\infty^2 + C(p\mu)^3 A^2 t^{-1}]\int \phi^{2p'} f^p.$$

This proves lemma 2.2.

Now given $0<\tau<\tau'<T$, let

$$\psi(t) = \begin{cases} 0, & 0\le t\le \tau, \\ \dfrac{t-\tau}{\tau'-\tau}, & \tau\le t\le \tau', \\ 1, & \tau'\le t\le T \end{cases}$$

Multiplying (2.4) by ψ, and noticing that $p'+1\le p^2$,

$$\frac{\partial}{\partial t}\left(\psi\int \phi^{2p'} f^p\right) + \psi\int |\nabla(\phi^{p'+1} f^{\frac{p}{2}})|^2$$
$$\le [\ p^6\hat{C}(t)\psi + |\psi'|\ ]\int \phi^{2p'} f^p,$$

where $\hat{C}(t) = C\|\nabla\phi\|_\infty^2 + C\mu^3 A^2 t^{-1}$. Integrating this with respect to t, we obtain

Lemma 2.3.

$$\int \phi^{2p'} f^p + \int_{\tau'}^{t}\int |\nabla(\phi^{p'+1} f^{\frac{p}{2}})|^2 \le \left(p^6\hat{C}(\tau') + \frac{1}{\tau'-\tau}\right)\int_\tau^T\int \phi^{2p'} f^p,\ \ \tau'\le t\le T.$$

Given $p\ge p'\ge p_0>1,\ 0\le\tau<T$, denote

$$H(p,p',\tau) = \int_\tau^T\int_{B_0} \phi^{2p'} f^p.$$

Lemma 2.4. *Given $p\ge p_0,\ 0\le\tau<\tau'<T$,*

$$H\left(\frac{3}{2}p, \frac{3}{2}p'+1, \tau'\right) \le AC[\ (\tau'-\tau)^{-1} + p^6\hat{C}(\tau')\ ]^{\frac{3}{2}} H(p,p',\tau)^{\frac{3}{2}}.$$

Proof: By Hölder, Sobolev inequalities,

$$
\begin{aligned}
H\left(\frac{3}{2}p, \frac{3}{2}p'+1, \tau'\right) &= \int_{\tau'}^{T}\int \phi^2(\phi^{2p'}f^p)^{\frac{3}{2}} \\
&\le \int_{\tau'}^{T}\left(\int \phi^{2p'}f^p\right)^{\frac{1}{2}}\left(\int \phi^{4p'+4}f^{2p}\right)^{\frac{1}{2}} \mathrm{d}\,t \\
&\le \left(\sup_{\tau'\le t\le T}\int \phi^{2p'}f^p\right)^{\frac{1}{2}}\ A\ \int_{\tau'}^{T}\int |\nabla(\phi^{p'+1}f^{\frac{p}{2}})|^2\,\mathrm{d}\,t.
\end{aligned}
$$

Applying Lemma 2.3, we obtain the desired estimate. □

Theorem 2.5. *Let f and u be non-negative functions on $B_0\times[0,T]$, such that $\frac{\partial}{\partial t}\mathrm{d}\,V_g \le c\phi^2 u\,\mathrm{d}\,V_g$ for some constant c, and*

$$
\frac{\partial f}{\partial t} \le \phi^2(\triangle f + uf) + 2a\phi|\nabla\phi||\nabla f| + b(|\nabla\phi|^2 - \phi\nabla\phi)f,\ \ 0\le t\le T.
$$

Assume that

$$
\left(\int_{B_0}\phi^2u^3\right)^{\frac{1}{3}} \le \mu t^{-\frac{1}{3}}.
$$

Then given $(x,t)\in B_0\times[0,T]$, $p_0>2$,

$$
|\phi(x)^2 f(x,t)| \le CA^{\frac{2}{p_0}}[\ \|\nabla\phi\|_\infty^2 + t^{-1}(1+A^2\mu^3)\ ]^{\frac{3}{p_0}}\left(\int_0^t\int_{B_0}\phi^{2p_0-4}f^{p_0}\right)^{\frac{1}{p_0}},
$$

where C depends on p_0, a and b.

Proof: Denote $\nu = \frac{3}{2}$*and* $\eta=\nu^6$. Fix $0<t<T$, and set

$$
\begin{aligned}
p'_k &= (p_0-2)\,\nu^k + \sum_{j=0}^{k-1}\nu^j, \\
p_k &= p_0\nu^k, \\
\tau_k &= t(1-\eta^{-k}), \\
\Phi_k &= H(p_k, p'_k, \tau_k)^{\frac{1}{p_k}}.
\end{aligned}
$$

Applying Lemma 2.4,

$$
H(p_{k+1}, p'_{k+1}, \tau_{k+1}) \le AC\left[\ \|\nabla\phi\|_\infty^2 + (1+\mu^3A^2)\frac{\eta}{\eta-1}t^{-1}\ \right]^\nu \eta^{k\nu}H(p_k,p'_k,\tau_k)^\nu.
$$

Therefore,

$$\Phi_{k+1} \le (AC)^{\frac{\sigma_{k+1}-1}{p_0}} \left(\|\nabla\phi\|_\infty^2 + (1+\mu^3 A^2)^{\frac{\eta}{\eta-1}} t^{-1} \right)^{\frac{\sigma_k}{p_0}} \cdot \eta^{\frac{\sigma_k'}{p_0}} H(p_0, p_0-2, 0)^{\frac{1}{p_0}},$$

where $\sigma_k = \sum_{i=0}^k \nu^{-i}$, $\sigma_k' = \sum_{i=0}^k i\nu^{-i}$. Letting $k \to \infty$, we obtain

$$|\phi^2 f(x,t)| \le CA^{\frac{2}{p_0}} \left[\ \|\nabla\phi\|_\infty^2 + t^{-1}(1+\mu^3 A^2) \ \right]^{\frac{3}{p_0}} \left(\int_0^T \int \phi^{2p_0-4} f^{p_0} \right)^{\frac{1}{p_0}}.$$

Now let $T \to t$. This proves the theorem. $\square$

Theorem 2.6. *Let $f \ge 0$. Solve*

$$\frac{\partial f}{\partial t} \le \phi^2(\triangle f + C_0 f^2) + 2a\phi|\nabla\phi||\nabla f| + b(|\nabla\phi|^2 - 2\phi\triangle\phi)f, \ 0 \le t \le T, \tag{2.6}$$

on $B_0 \times [0,T]$. Assume that

$$\frac{\partial}{\partial t} \mathrm{d}\, V_g \le C\phi^2 f \,\mathrm{d}\, V_g$$

and that

$$\left(\int_{B_0} f_0^2 \right)^{\frac{1}{2}} \le (5eC_0 A)^{-1},$$

where $f_0(x) = f(x,0)$. Then

$$|\phi^2(x) f(x,t)| \le CA(t\|\nabla\phi\|_\infty^2 + 1)^2 t^{-1},$$

where $0 < t < \min(T, \|\nabla\phi\|_\infty^{-2})$, $C = C(C_0, a, b)$.

Proof: Let $[0,T'] \subset [0,T]$ be the maximal interval such that

$$e_0 = \sup_{0 \le t \le T'} \left(\int_{B_0} f^2 \right)^{\frac{1}{2}} \le (4C_0 A)^{-1}.$$

Applying Lemma 2.1 to (2.6), we have, for $0 \le t \le T'$,

$$\begin{aligned} \frac{\partial}{\partial t} \int f^p + 2\left(1 - \frac{1}{p}\right)^2 \int |\nabla(\phi f^{\frac{p}{2}})|^2 \\ \le p \int |\nabla\phi|^2 f^p + pC_0 A \left(\int f^2 \right)^{\frac{1}{2}} \int |\nabla(\phi f^{\frac{p}{2}})|^2. \end{aligned}$$

Therefore, for $p = 2$, the bound on the L^2 norm of f implies that for $0 \le t \le T'$,

$$\frac{\partial}{\partial t} \int f^2 \le 2\|\nabla\phi\|_\infty \int f^2,$$

which implies that

$$\int f^2 \le e^{2\|\nabla\phi\|^2 t} \int f_0^2.$$

In particular, if $T' < \|\nabla\phi\|^{-2}$, then

$$e_0 \le e \int f_0^{\frac{n}{2}} \le (5C_0A)^{-1} < (4C_0A)^{-1}.$$

This contradicts the assumed maximality of $[0, T']$. We can therefore assume that $T' \ge \min((\log 2)\|\nabla\phi\|^{-2}, T)$.

By the same argument of Lemma 2.2, we have an estimate of the form

$$\int f^p \cdot \int_0^t \int |\nabla(\phi f^{\frac{p}{2}})|^2 \le C(t^{-1} + \|\nabla\phi\|_\infty) \int_0^t \int f^p.$$

Therefore,

$$\begin{aligned}
\int \phi^2 f^3 \le & C(t^{-1} + \|\nabla\phi\|_\infty) \int_0^t \int \phi^2 f^3 \\
\le & C(t^{-1} + \|\nabla\phi\|_\infty) \int_0^t \left(\int f^2\right)^{\frac{1}{2}} \left(\int (\phi f)^4\right)^{\frac{1}{2}} \mathrm{d}\, t \\
\le & Ce_0A(t^{-1} + \|\nabla\phi\|_\infty) \int_0^t \int |\nabla(\phi f)|^2 \\
\le & Ce_0A(t^{-1} + \|\nabla\phi\|_\infty)^2 \int_0^t \int f^2 \\
\le & CAt(t^{-1} + \|\nabla\phi\|_\infty)^2 e_0^3.
\end{aligned}$$

Set

$$\mu^3 = CA(1 + t\|\nabla\phi\|_\infty^2)^2 e^3$$

and notice that Theorem 2.5 still holds, when $p_0 \to 2$. We then obtain the desired estimate. □

The argument also implies the following

Corollary 2.7. *Let f satisfy the assumptions of Theorem 2.6. Then given $u \ge 0$ such that*

$$\frac{\partial u}{\partial t} \le \phi^2(\triangle u + c_0 f u) + a \cdot \nabla u + bu,$$

the following estimate holds for $0 \le t < \min(T, (\log 2)\|\nabla\phi\|_\infty^{-2})$,

$$|\phi(x)^2 u(x,t)| \le CA^{\frac{2}{p_0}}(1 + t\|\nabla\phi\|_\infty^2)^2 t^{-\frac{2}{p_0}} \left(\int_{B_0} u^{p_0}\right)^{\frac{1}{p_0}},$$

where $u_0(x,t) = u(x,0)$, and C depends on p_0, a and b.

3 Existence of Local Ricci Flow

Let X be a smooth 4-manifold without boundary. Given a smooth Riemannian metric g_0 and a smooth compactly supported function ϕ, we want to study the following evolution equation

$$\begin{cases} \dfrac{\partial g}{\partial t} & = -2\phi^2 \operatorname{Ric}(g), \\ g(0) & = g_0. \end{cases} \tag{3.1}$$

Theorem 3.1. *There exists $T > 0$ such that (3.1) has a smooth solution for $0 \le t \le T$.*

Proof: Given $\varepsilon > 0$, consider

$$\begin{cases} \dfrac{\partial g}{\partial t} & = -2(\varepsilon^2 + \phi^2) \operatorname{Ric}(g), \\ g(0) & = g_0. \end{cases} \tag{3.2}$$

We want to use DeTurck's trick so that this system can be reduced to a nonlinear, strictly parabolic system. First, we fix a metric $\hat{g}$ on X. Let Γ_{ij}^k and $\hat{\Gamma}_{ij}^k$ denote the Christoffel symbols of g and $\hat{g}$ respectively. Our aim is to give an expression of $\operatorname{Ric}(g) - \operatorname{Ric}(\hat{g})$. By direct calculation, we have

$$\Gamma_{jk}^i - \hat{\Gamma}_{jk}^i = \frac{1}{2} g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l}),$$

where $g_{jl,k} = \dfrac{\partial g_{jl}}{\partial x^k} - g_{sl}\hat{\Gamma}_{kj}^s - g_{js}\hat{\Gamma}_{kl}^s$, the covariant derivative with respect to the metric $\hat{g}$.

Recall that in local coordinates

$$R_{ijl}^p = \frac{\partial}{\partial x^i}\Gamma_{jl}^p - \frac{\partial}{\partial x^j}\Gamma_{il}^p + \Gamma_{ig}^p\Gamma_{jl}^q - \Gamma_{jq}^p\Gamma_{il}^q,$$

and

$$R_{ik} = g^{jl} g_{hk} R_{jil}^h.$$

So

$$\begin{aligned} R_{ij} - \hat{R}_{ij} =& g^{kl} g_{hj} \left(\frac{\partial}{\partial x^i}\Gamma_{kl}^h - \frac{\partial}{\partial x^k}\Gamma_{il}^h \right) + \text{other terms} \\ =& g^{kl} g_{hj} \left(\frac{\partial}{\partial x^i}(\Gamma_{kl}^h - \hat{\Gamma}_{kl}^h) - \frac{\partial}{\partial x^k}(\Gamma_{il}^h - \hat{\Gamma}_{il}^h) \right) + \text{other terms} \\ =& -\frac{1}{2} g^{kl} g_{ij,kl} + \frac{1}{2} g^{kl}(g_{il,jk} + g_{jl,ik} - g_{kl,ij}) + \text{other terms.} \end{aligned}$$

Now we set

$$X^p = -g^{pi} g^{kl}(g_{ik,l} - \frac{1}{2} g_{kl,i})$$

and $X = X^p \dfrac{\partial}{\partial x^p}$, then

$$\begin{aligned}
(L_X g)_{kl} =& (L_X g)\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) \\
=& L_X\left(g\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)\right) - g\left(L_X \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) - g\left(\frac{\partial}{\partial x^k}, L_X \frac{\partial}{\partial x^l}\right) \\
=& X\left(g\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)\right) - g\left(\nabla_X \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^k}} X, \frac{\partial}{\partial x^l}\right) \\
& - g\left(\frac{\partial}{\partial x^k}, \nabla_X \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^l}} X\right) \\
=& g\left(\nabla_{\frac{\partial}{\partial x^k}} X, \frac{\partial}{\partial x^l}\right) + g\left(\frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^l}} X\right) \\
=& \frac{\partial X^p}{\partial x^k} g\left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^l}\right) + X^p g\left(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^l}\right) + \frac{\partial X^p}{\partial x^l} g\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^p}\right) \\
& + X^p\left(\frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^l}} \frac{\partial}{\partial x^p}\right) \\
=& \frac{\partial X^p}{\partial x^k} g_{pl} + \frac{\partial X^p}{\partial x^l} g_{kp} + X^p \frac{\partial}{\partial x^p} g_{kl}.
\end{aligned}$$

Thus

$$R_{ij} - \hat{R}_{ij} = -\frac{1}{2} g^{kl} g_{ij,kl} - \frac{1}{2}(L_X g)_{ij} + \text{other terms.} \tag{3.3}$$

We set

$$(F(g))_{ij} = g^{kl} g_{ij,kl} + Q_{ij},$$

where Q_{ij} involves the other terms in (3.3), then

$$\mathrm{Ric}(g) - \mathrm{Ric}(\hat{g}) = -\frac{1}{2} F(g) - \frac{1}{2} L_X g.$$

We define a one-parameter diffeomorphism group $\Phi_t : X \to X$ as follows.

$$\begin{cases} \dfrac{\mathrm{d}\,\Phi_t(x)}{\mathrm{d}\,t} = [(\phi \circ \Phi_t^{-1})^2 + \varepsilon^2] X(t, \Phi_t(x)), \\ \Phi_0(x) = x, \end{cases}$$

where $X = X^p \dfrac{\partial}{\partial x^p}$ given as above.

Consider the following initial value problem

$$\begin{cases} \dfrac{\partial \bar{g}}{\partial t} &= [(\phi \circ \Phi_t^{-1})^2 + \varepsilon^2][F(\bar{g}) - 2\operatorname{Ric}(\hat{g})] - P, \\ \bar{g}(0) &= g_0, \end{cases}$$

where $P_{ij} = \frac{\partial}{\partial x^i}[(\phi \circ \Phi_t^{-1})^2]\bar{g}(X, \frac{\partial}{\partial x^j}) + \frac{\partial}{\partial x^j}[(\phi \circ \Phi_t^{-1})^2]\bar{g}(X, \frac{\partial}{\partial x^i})$. Then a direct calculation shows that $g = \Phi_t^*(\bar{g})$ is the solution of (3.2). Indeed,

$$\begin{aligned} \frac{\partial g}{\partial t} =& \frac{\partial}{\partial t}\Phi_t^*(\bar{g}) \\ =& \Phi_t^*\left(\frac{\partial \bar{g}}{\partial t}\right) + \Phi_t^*(L_{\Phi_t} \cdot \bar{g}) \\ =& \Phi_t^*\left(\frac{\partial \bar{g}}{\partial t} + L_{[(\phi\circ\Phi_t^{-1})^2+\varepsilon^2]X}\bar{g}\right) \\ =& \Phi_t^*\{[(\phi \circ \Phi_t^{-1})^2 + \varepsilon^2][F(\bar{g}) - 2\operatorname{Ric}(\hat{g}) + L_X\bar{g}]\} \\ =& -2\Phi_t^*\{[(\phi \circ \Phi_t^{-1})^2 + \varepsilon^2]\operatorname{Ric}\bar{g}\} \\ =& -2(\phi^2 + \varepsilon^2)\operatorname{Ric}(g), \end{aligned}$$

where we used the following fact,

$$(L_{fX}g)_{ij} = f(L_Xg)_{ij} + \left(\frac{\partial}{\partial x^i}f\right)g\left(X, \frac{\partial}{\partial x^j}\right) + \left(\frac{\partial}{\partial x^j}f\right)g\left(X, \frac{\partial}{\partial x^i}\right).$$

For our purposes, we may in addition assume that the curvature and the Ricci tensors of the initial metric admit a local L^2 and L^p norm bounds respectively, where $p > 2$. By the argument in the next section, we can then show that the curvature and its covariant derivative satisfy a local heat equation. Also, they can be shown to satisfy L^2 energy bounds that are independent of $\varepsilon > 0$. So (3.2) has a solution for some time interval $[0, T)$, where T is independent of ε. Thus as $\varepsilon \to 0$, the solution of (3.2) converges to a solution of (3.1).

4 Smoothing a Riemannian Metric

Let X be a smooth manifold with Riemannian metric g_0 and Ω an open subset of X. Let ϕ be a nonnegative smooth compactly supported function on Ω. Consider the following

evolution equation

$$\begin{cases} \dfrac{\partial g}{\partial t} & = -2\phi^2 \operatorname{Ric}(g), \\ g(0) & = g_0. \end{cases} \tag{4.1}$$

It is easy to check that the curvature tensor Rm and Ricci tensor Ric satisfy the following equations respectively,

$$\frac{\partial \operatorname{Rm}}{\partial t} = \phi^2(\triangle \operatorname{Rm} + Q(\operatorname{Rm}, \operatorname{Rm})) + 2\phi a(\nabla\phi, \nabla \operatorname{Rm}) + b(\nabla\phi, \nabla\phi, \operatorname{Rm}) + \phi c(\nabla^2\phi, \operatorname{Rm})$$

and

$$\frac{\partial \operatorname{Ric}}{\partial t} = \phi^2(\triangle \operatorname{Ric} + Q(\operatorname{Rm}, \operatorname{Ric})) + 2\phi a(\nabla\phi, \nabla \operatorname{Ric}) + b(\nabla\phi, \nabla\phi, \operatorname{Ric}) + \phi c(\nabla^2\phi, \operatorname{Ric}).$$

Notice that $\phi \in C_0^\infty(\Omega)$, we then have constant $c_1,\ c_2,\ ,c_3 > 0$ such that

$$\phi c(\nabla^2\phi, \operatorname{Rm}) \leq -c_1\phi\triangle\phi|\operatorname{Rm}| + c_2|\nabla\phi|^2|\operatorname{Rm}| + c_3\phi|\nabla\phi||\operatorname{Rm}|$$

and

$$\phi c(\nabla^2\phi, \operatorname{Ric}) \leq -c_1\phi\triangle\phi|\operatorname{Ric}| + c_2|\nabla\phi|^2|\operatorname{Ric}| + c_3\phi|\nabla\phi||\operatorname{Ric}|.$$

Then a direct calculation gives

$$\frac{\partial|\operatorname{Rm}|}{\partial t} \leq \phi^2(\triangle|\operatorname{Rm}| + c_0|\operatorname{Rm}|^2) + 2a\phi|\nabla\phi||\nabla \operatorname{Rm}| + b(|\nabla\phi|^2 - \phi\triangle\phi)|\operatorname{Rm}|, \tag{4.2}$$

and

$$\frac{\partial|\operatorname{Ric}|}{\partial t} \leq \phi^2(\triangle|\operatorname{Ric}| + c_0|\operatorname{Rm}||\operatorname{Ric}|) + 2a\phi|\nabla\phi||\nabla \operatorname{Ric}| + b(|\nabla\phi|^2 - \phi\triangle\phi)|\operatorname{Ric}|. \tag{4.3}$$

Again the results in this section are due to D. Yang [5].

Theorem 4.1. *There exist constant C_1 and C_2 such that if*

$$\left(\int_\Omega |\operatorname{Rm}(g_0)|^2 \,\mathrm{d}\,V_{g_0}\right)^{\frac{1}{2}} \leq [\ C_1 C_s(\Omega)\ ]^{-1}$$

and for any $p > 2$,

$$\left(\int_\Omega |\operatorname{Ric}(g_0)|^p \,\mathrm{d}\,V_{g_0}\right)^{\frac{1}{p}} < K,$$

then the equation (4.1) has a smooth solution for $t \in [0, T)$, where

$$T \geq \min\left(\|\nabla\phi\|_\infty^{-2}, C_2 K^{-\frac{p}{p-2}} C_s(\Omega)^{-\frac{2}{p-2}}\right).$$

Moreover, for $t \in (0,T)$, the Riemannian curvature tensor satisfies the following bound,

$$\|\phi^2 \operatorname{Rm}\|_\infty \le C_3 C_s(\Omega)(t\|\nabla\phi\|_\infty^2 + 1)t^{-1}. \tag{4.4}$$

Here C_1 and C_3 only depend on the dimension of X; C_2 depends on the dimension of X and p.

Proof: By Theorem 3.1, the equation (4.1) has a smooth solution on a sufficiently small time interval starting at $t = 0$. Let $[0, T_{\max})$ be a maximal time interval on which (4.1) has a smooth solution and such that the following hold for each metric $g(t)$,

$$\|f\|_\psi^2 \le \psi A_0 \|\nabla f\|_2^2,\ f \in C_0^\infty(\Omega); \tag{4.5}$$

$$\frac{1}{2} g_0 \le g(t) \le 2g_0; \tag{4.6}$$

$$\|\operatorname{Rm}(g(t))\|_2 \le 2(C_1 A_0)^{-1}. \tag{4.7}$$

Suppose that $T_{\max} < T_0 = \min(\|\nabla\phi\|_\infty^{-2},\ C_2 K^{-\frac{p}{p-2}} A^{-\frac{2}{p-2}})$. We will show that this leads to a contradiction.

First, notice that the curvature tensor Rm satisfies (4.2), then according to the proof of Theorem 2.6, we have

$$\|\operatorname{Rm}(g(t))\|_2 < e\|\operatorname{Rm}(g_0)\|_2 \le 2e[C(n)4A_0]^{-1} < 2[C(n)A_0]^{-1},$$

which implies a strict inequality for (2.14).

Next, since the Ricci curvature Ric satisfies (4.3), then Corollary 2.7 implies that

$$|\phi^2 \operatorname{Ric}(g(t))| \le C_2 A_0^{\frac{2}{p}} (1 + t\|\nabla\phi\|_\infty^2)^2 t^{-\frac{2}{p}} K.$$

Applying the bound on Ric to the following

$$\left| \frac{\mathrm{d}}{\mathrm{d}\,t} \int f^p \,\mathrm{d}\, V_g \right| \le 2\|\phi^2 \operatorname{Ric}\|_\infty \int f^p \,\mathrm{d}\, V_g,$$

we have

$$\log \frac{\|f\|_p(t)}{\|f\|_p(0)} < \log 2.$$

The differential inequality

$$\left| \frac{\mathrm{d}}{\mathrm{d}\,t} \int |\nabla f|^2 \,\mathrm{d}\, V_g \right| \le 2\|\operatorname{Ric}\|_\infty \int |\nabla f|^2 \,\mathrm{d}\, V_g$$

leads to an analogous estimate. Therefore, it follows that for any $t \leq T_0$,

$$\|f\|_4^2(t) < 2\|f\|_4^2(0) \leq 2A_0\|\nabla f\|_2^2(0) < 4A_0\|\nabla f\|_2^2(t),$$

that is to say (4.5) holds with strict inequality.

To show that (4.6) holds with strict inequality, we use Hamilton's trick. Simply fix a tangent vector v with respect to $g(t)$, then

$$\frac{\mathrm{d}}{\mathrm{d}\,t}|v|^2_{g(t)} = \frac{\mathrm{d}}{\mathrm{d}\,t}(g_{ij}(t)v^i v^j) = g'_{ij}(t)v^i v^j$$

implies

$$\left|\frac{\mathrm{d}}{\mathrm{d}\,t}\log|v|^2_{g(t)}\right| \leq |g'_{ij}(t)| \leq 2\phi^2|\operatorname{Ric}|.$$

So for $0 \leq t \leq T_2 < T_0$,

$$\log\frac{|v|^2_{g(t)}}{|v|^2_{g(0)}} \leq \int_0^{T_2}|g'_{ij}(t)|\,\mathrm{d}\,t \leq 2\|\phi^2\operatorname{Ric}\|_\infty T_2 < \log 2,$$

which implies

$$\frac{1}{2}|v|^2_{g(0)} < |v|^2_{g(t)} < 2|v|^2_{g(0)},$$

for $t < T_0$.

Finally, by differentiating the evolution equation for Rm, we see that the covariant derivatives of Rm satisfy evolution equations for which L^2 energy bounds can be obtained. Therefore we can use Hamilton's argument in §14 of [3] to show that $g(t)$ has a smooth limit as $t \to T_{\max}$. If $T_{\max} < T_0$, we would be able to extend the solution to (4.1) smoothly beyond $T_{\max}$ with (4.5), (4.6) and (4.7) still holding. This contradicts the assumed maximality of $T_{\max}$. Hence, we conclude that $T_{\max} \geq T_0$.

The estimate (4.4) follows from Theorem 2.6. □

5 Local Volume Estimate

We consider more generally any system of the type

$$\triangle\operatorname{Ric} = \operatorname{Rm} * \operatorname{Ric} + \mathrm{B}, \tag{5.1}$$

where B is the Bach tensor. Recall that

$$B_{ij} = 2\nabla^k\nabla^l W^+_{ikjl} + R^{kl}W^+_{ikjl}.$$

We assume the following local Sobolev inequality,

$$\|f\|^2_{L^4(\Omega)} \le C_s(\Omega)\|\nabla f\|_2^2,\ \forall f \in C_0^\infty(\Omega).$$

Lemma 5.1. *There exist constant* ε, C *such that if* $\|\operatorname{Rm}\|_{L^2(B(p,r))} \le \varepsilon$ *and* $\mathrm{B} \in L^2(B(p,r))$, *then*

$$\left\{\int_{B(p,\frac{r}{2})} |\operatorname{Ric}|^4 \,\mathrm{d}V_g\right\}^{\frac12} \le \frac{C}{r^2}\left(\int_{B(p,r)} |\operatorname{Ric}|^2 \,\mathrm{d}V_g\right) + C\left(\int_{B(p,r)} |\operatorname{Ric}|^2\right)^{\frac12}\left(\int_{B(p,r)} |B|^2\right)^{\frac12}.$$

Proof: From (5.1), it follows that

$$\triangle|\operatorname{Ric}| \ge -|\operatorname{Rm}||\operatorname{Ric}| - |\mathrm{B}|.$$

We may assume that $r = 1$. The lemma then follows by scaling the metric. Let $0 \le \phi \le 1$ be a function supported in $B(p,1)$, then

$$\begin{aligned}\int_{B(p,1)} \phi^2|\operatorname{Ric}|^2|\operatorname{Rm}| \ge& \int \phi^2|\operatorname{Ric}|(-\triangle|\operatorname{Ric}| - \mathrm{B})\\ =& \int \nabla(\phi^2|\operatorname{Ric}|)\cdot\nabla|\operatorname{Ric}| - \int \phi^2|\operatorname{Ric}||\mathrm{B}|\\ \ge& -\delta^{-1}\int |\nabla\phi|^2|\operatorname{Ric}|^2 + (1-\delta)\int |\phi\nabla|\operatorname{Ric}||^2\\ &-\left(\int \phi^2|\operatorname{Ric}|^2\right)^{\frac12}\left(\int \phi^2|\mathrm{B}|^2\right)^{\frac12}.\end{aligned}$$

Next, using the Sobolev constant bound, we have

$$\left(\int (\phi|\operatorname{Ric}|)^4\right)^{\frac12} \le C\int |\nabla(\phi|\operatorname{Ric}|)|^2 \le C\int |\nabla\phi|^2|\operatorname{Ric}|^2 + C\int \phi^2|\nabla|\operatorname{Ric}||^2.$$

Choosing δ sufficiently small yields

$$\begin{aligned}\left(\int (\phi|\operatorname{Ric}|)^4\right)^{\frac12} \le& C\int \phi^2|\operatorname{Ric}|^2|\operatorname{Rm}| + C\int |\nabla\phi|^2|\operatorname{Ric}|^2\\ &+ C\left(\int \phi^2|\operatorname{Ric}|^2\right)^{\frac12}\left(\int \phi^2|\mathrm{B}|^2\right)^{\frac12}\\ \le& C\left(\int \phi^2|\operatorname{Rm}|^2\right)^{\frac12}\left(\int \phi^2|\operatorname{Ric}|^4\right)^{\frac12} + C\int |\nabla\phi|^2|\operatorname{Ric}|^2\\ &+ C\left(\int \phi^2|\operatorname{Ric}|^2\right)^{\frac12}\left(\int \phi^2|\mathrm{B}|^2\right)^{\frac12}.\end{aligned}$$

Therefore, for ε sufficiently small, we have

$$\left(\int \phi^2 |\operatorname{Ric}|^4\right)^{\frac{1}{2}} \leq C\int |\operatorname{Ric}|^2 + C\left(\int |\operatorname{Ric}|^2\right)^{\frac{1}{2}}\left(\int |\mathrm{B}|^2\right)^{\frac{1}{2}}.$$

We then choose the cut-off function ϕ such that $\phi \equiv 1$ in $B(p, \frac{1}{2})$, $\phi = 0$ for $r = 1$, $|\nabla\phi| \leq C$, and we have

$$\left(\int_{B(p,\frac{1}{2})} |\operatorname{Ric}|^4\right)^{\frac{1}{2}} \leq C\int_{B(p,1)} |\operatorname{Ric}|^2 + C\left(\int |\operatorname{Ric}|^2\right)^{\frac{1}{2}}\left(\int |\mathrm{B}|^2\right)^{\frac{1}{2}}.$$

Scaling the metric, we obtain the lemma. □

Lemma 5.2. *With the same assumption of Lemma 5.1, we have*

$$\begin{aligned}\left(\int_{B(p,\frac{r}{4})} |\operatorname{Rm}|^4 \, \mathrm{d}V_g\right)^{\frac{1}{2}} \leq & C\left(\int_{B(p,r)} |\operatorname{Ric}|^2\right)^{\frac{1}{2}}\left(\int_{B(p,r)} |\mathrm{B}|^2\right)^{\frac{1}{2}} \\ & + \frac{C}{r^2}\int_{B(p,r)} |\operatorname{Rm}|^2.\end{aligned}$$

Proof: Again we may assume that $r = 1$. Let ϕ be a cut-off function in $B(p, 1)$, such that, $\phi \equiv 1$ in $B(p, \frac{1}{2})$ and $|\nabla\phi| \leq C$. We have, by lemma 5.1,

$$\begin{aligned}\int_{B(p,1)} \phi^2 |\nabla \operatorname{Ric}|^2 =& -\int \phi^2 \langle \triangle \operatorname{Ric}, \operatorname{Ric}\rangle - 2\int \phi\langle \nabla \operatorname{Ric}, \nabla\phi \cdot \operatorname{Ric}\rangle \\ =& -\int \phi^2\langle \operatorname{Rm} * \operatorname{Ric}, \operatorname{Ric}\rangle - \int \phi^2\langle \mathrm{B}, \operatorname{Ric}\rangle - 2\int \phi\langle \nabla \operatorname{Ric}, \nabla\phi \cdot \operatorname{Ric}\rangle \\ \leq& C\left(\int \phi^2 |\operatorname{Rm}|^2\right)^{\frac{1}{2}}\left\{\int_{B(p,1)} |\operatorname{Ric}|^2 + \left(\int |\operatorname{Ric}|^2\right)^{\frac{1}{2}}\left(\int |\mathrm{B}|^2\right)^{\frac{1}{2}}\right\} \\ & + C\int |\operatorname{Ric}|^2 + C\delta\int \phi^2 |\nabla \operatorname{Ric}|^2 + C\left(\int |\mathrm{B}|^2\right)^{\frac{1}{2}}\left(\int |\operatorname{Ric}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

By choosing δ small and $\varepsilon < 1$, we have

$$\begin{aligned}\int_{B(p,\frac{1}{2})} |\nabla \operatorname{Ric}|^2 \leq & (1+\varepsilon)C\int_{B(p,1)} |\operatorname{Ric}|^2 \\ & + (1+\varepsilon)C\left(\int_{B(p,1)} |\operatorname{Ric}|^2\right)^{\frac{1}{2}}\left(\int_{B(p,1)} |\mathrm{B}|^2\right)^{\frac{1}{2}} \\ \leq & 2C\int_{B(p,1)} |\operatorname{Ric}|^2 \\ & + 2C\left(\int_{B(p,1)} |\operatorname{Ric}|^2\right)^{\frac{1}{2}}\left(\int_{B(p,1)} |\mathrm{B}|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{5.2}$$

Next, let ϕ be a cutoff function in $B(p, \frac{1}{2})$, such that $\phi \equiv 1$ in $B(p, \frac{1}{4})$ and $|\nabla \phi| \leq C$. Recall that

$$\triangle \operatorname{Rm} = L(\nabla^2 \operatorname{Ric}) + \operatorname{Rm} * \operatorname{Rm},$$

where $L(\nabla^2 \operatorname{Ric})$ denotes a linear expression in second derivatives of the Ricci tensor. We then have

$$\begin{aligned}\int_{B(p,\frac{1}{2})} \langle \triangle \operatorname{Rm}, \phi^2 \operatorname{Rm} \rangle &= \int \langle \nabla^2 \operatorname{Ric} + \operatorname{Rm} * \operatorname{Rm}, \phi^2 \operatorname{Rm} \rangle \\ &= - \int \langle 2\phi \nabla \operatorname{Ric}, (\nabla \phi) \operatorname{Rm} \rangle - \int \phi^2 \langle \nabla \operatorname{Ric}, \nabla \operatorname{Rm} \rangle \\ &\quad + \int \langle \operatorname{Rm} * \operatorname{Rm}, \phi^2 \operatorname{Rm} \rangle.\end{aligned}$$

This yields

$$\begin{aligned}\left| \int_{B(p,\frac{1}{2})} \langle \triangle \operatorname{Rm}, \phi^2 \operatorname{Rm} \rangle \right| \leq & C \int \phi^2 |\nabla \operatorname{Ric}|^2 + C \int |\nabla \phi|^2 \nabla \operatorname{Rm}|^2 \\ & + \frac{C}{\delta} \int \phi^2 |\nabla \operatorname{Ric}|^2 + C\delta \int \phi^2 |\nabla \operatorname{Rm}|^2 + C \int \phi^2 |\operatorname{Rm}|^3.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int_{B(p,\frac{1}{2})} \phi^2 |\nabla \operatorname{Rm}|^2 = & \int \langle 2\phi \nabla \operatorname{Rm}, (\nabla \phi) \operatorname{Rm} \rangle - \int \phi^2 \langle \triangle \operatorname{Rm}, \operatorname{Rm} \rangle \\ \leq & \frac{C}{\delta} \int |\nabla \phi|^2 |\operatorname{Rm}|^2 + 2C\delta \int \phi^2 |\nabla \operatorname{Rm}|^2 + C \int |\nabla \phi|^2 |\operatorname{Rm}|^2 \\ & + \frac{C'}{\delta} \int \phi^2 |\nabla \operatorname{Ric}|^2 + C \int \phi^2 |\operatorname{Rm}|^3.\end{aligned}$$

Choosing δ sufficiently small and using (5.2), we obtain

$$\begin{aligned}\int \phi^2 |\nabla \operatorname{Rm}|^2 \leq & C \int |\nabla \phi|^2 |\operatorname{Rm}|^2 + C \int |\operatorname{Rm}|^2 + C \int \phi^2 |\operatorname{Rm}|^3 \\ & + C \left(\int |\operatorname{Ric}|^2 \right)^{\frac{1}{2}} \left(\int |\mathrm{B}|^2 \right)^{\frac{1}{2}}.\end{aligned}$$

Using the Sobolev inequality,

$$\begin{aligned}\left(\int|\phi\operatorname{Rm}|^4\right)^{\frac{1}{2}} \leq & C\int|\nabla|\phi\operatorname{Rm}||^2\\ \leq & C\int|\nabla\phi|^2|\operatorname{Rm}|^2+C\int\phi^2|\nabla|\operatorname{Rm}||^2\\ \leq & C\int|\operatorname{Rm}|^2+C\left(\int|\operatorname{Rm}|^2\right)^{\frac{1}{2}}\left(\int\phi^4|\operatorname{Rm}|^4\right)^{\frac{1}{2}}\\ & +C\left(\int|\operatorname{Ric}|^2\right)^{\frac{1}{2}}\left(\int|\mathrm{B}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore by choosing ε small, we obtain

$$\left(\int_{B(p,\frac{1}{4})}|\operatorname{Rm}|^4\right)^{\frac{1}{2}} \leq C\int|\operatorname{Rm}|^2+C\left(\int|\operatorname{Ric}|^2\right)^{\frac{1}{2}}\left(\int|\mathrm{B}|^2\right)^{\frac{1}{2}}.$$

Scaling the metric, we obtain the lemma. □

Theorem 5.3. *Assume that (5.1) is satisfied. Let $B(p,r)$ be a geodesic ball around the point p. Then there exist constants ε_0, C (depending on the Sobolev constant $C_s(B(p,r))$) such that if*

$$\|\operatorname{Rm}\|_{L^2(B(p,2r))} \leq \varepsilon_0$$

and $\mathrm{B} \in L^2(B(p,2r))$*, then*

$$\operatorname{Vol}(B(p,r)) \leq Cr^4.$$

Proof: We assume that $r=1$. By lemma 5.1 and 5.2, we have

$$\int|\operatorname{Rm}|^3 \leq \left(\int|\operatorname{Rm}|^2\right)^{\frac{1}{2}}\left(\int|\operatorname{Rm}|^4\right)^{\frac{1}{2}} \leq C.$$

Then for ε_0 suitably chosen, by Theorem 4.1, the local Ricci flow

$$\begin{cases}\dfrac{\partial g(t)}{\partial t} & = -2\phi^2\operatorname{Ric}(g(t)),\\ g(0) & = g\end{cases}$$

has a smooth solution for $t\in[0,T)$, where

$$T \geq \min(\|\nabla\phi\|_\infty^{-2}, C^{-3}C_s^{-2})$$

and for $t \in (0, T)$, the Riemannian curvature tensor satisfies the following bound

$$\|\phi^2 \operatorname{Rm}\|_\infty \leq C C_s (t \|\nabla \phi\|_\infty^2 + 1) t^{-1}.$$

Therefore,

$$\operatorname{Vol}_{g(t)}(B(p, 1)) \leq C$$

for any fixed $t \in (0, T)$. Since we can find a constant C such that

$$\frac{1}{C} g \leq g(t) \leq C g,$$

then for the metric g we still obtain the volume estimate

$$\operatorname{Vol}_g(B(p, 1)) \leq C.$$

This proves the theorem. $\square$

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